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FINDING A HOMOMORPHISM BETWEEN
TWO WORDS IS NP-COMPLETE

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- WORKING PAPER -

Abstract. We demonstrate that to find a homomorphism between two words x and y where letters of x can be chosen from an infinite alphabet and y is a word over a two letter alphabet is an NP-complete problem.

Looking for NP-complete problems today forms an active research area within complexity theory (see, e.g., [1]). Showing that a problem is NP-complete often contributes to our understanding of the difficulty of the problem and of the nature of NP-complete problems in general.

In this note we demonstrate that a very common problem from formal language theory is NP-complete: finding a nonerasing homomorphism between two words x and y (providing we do not restrict a priori the size of the alphabet of the word x ; y can be chosen over a two-letter alphabet.) The problem remains NP-complete even when arbitrary homomorphisms are admitted. It may very well be one of the simplest NP-complete problems known. On the other hand one can easily see that if the size of the alphabet is limited a priori then the problem is in P .

Formally it is defined as follows. (In the sequel, given a word x , $\text{alph } x$ denotes the set of letters appearing in x and $\#_a x$ denotes the number of occurrences of the letter a in x ; $\text{HOM}(\Sigma_1, \Sigma_2)$ denotes the class of nonerasing homomorphisms from Σ_1^* into Σ_2^*).

Let Σ be an infinite alphabet, Δ its subset containing two elements, $\Delta = \{b, c\}$ say. Let

$\text{MATCH}(\Sigma, \Delta) = \{(x, y) : \text{alph } x \subseteq \Sigma, \text{alph } y \subseteq \Delta \text{ and there exists a homomorphism } h \text{ in } \text{HOM}(\text{alph } x, \text{alph } y) \text{ such that } h(x) = y\}.$

Theorem. Membership in $\text{MATCH}(\Sigma, \Delta)$ is NP-complete.

Proof.

(i) Obviously membership in $\text{MATCH}(\Sigma, \Delta)$ is in NP.

(ii) Since 3-satisfiability is NP-complete (see, e.g., [1]) it suffices to show that for every Boolean expression Ψ in 3-conjunctive normal form

(3-CNF) there exist a word x_Ψ in Σ^+ and y_Ψ in Δ^+ such that

(*) $\dots (x_\Psi, y_\Psi) \in \text{MATCH}(\Sigma, \Delta)$ if and only if Ψ is satisfiable.

To this aim we proceed as follows.

Let $V = \{P_1, \dots, P_\ell\}$ be a set of Boolean variables and let

$$\Psi = (P'_{i_1} \vee P'_{j_1} \vee P'_{k_1}) \wedge \dots \wedge (P'_{i_n} \vee P'_{j_n} \vee P'_{k_n}),$$

with $i_q, j_q, k_q \in \{1, \dots, \ell\}$ be a Boolean expression over V in 3-CNF where,

for $q \in \{1, \dots, n\}$, each P'_{i_q} (P'_{j_q}, P'_{k_q} respectively) is either a variable

P_{i_q} (P_{j_q}, P_{k_q} respectively) or its negation \bar{P}_{i_q} ($\bar{P}_{j_q}, \bar{P}_{k_q}$ respectively).

Our construction of words x_Ψ, y_Ψ takes two steps.

STEP 1.

We will construct a finite set W of pairs of words (α, β) with $\alpha \in V_\Psi^+, V_\Psi \subset \Sigma$, and $\beta \in \Delta^+$ such that

(**) ... $\begin{cases} \Psi \text{ is satisfiable if and only if there exists a homomorphism} \\ h \text{ in } \text{HOM}(V_\Psi, \{b\}) \text{ such that, for every } (\alpha, \beta) \text{ in } W, h(\alpha) = \beta. \end{cases}$

Let $\bar{V} = \{\bar{P}_q : 1 \leq q \leq \ell\}$. Clearly we can assume that $V \cup \bar{V} \subseteq \Sigma$.

Let $T_1, \dots, T_n, U_1, \dots, U_n$ be new elements of Σ different from each other.

Let $V_\Psi = V \cup \bar{V} \cup \{T_q : 1 \leq q \leq n\} \cup \{U_q : 1 \leq q \leq n\}$ and let

$$W_1 = \{(P_q \bar{P}_q, b^3) : 1 \leq q \leq \ell\},$$

$$W_2 = \{(T_q P'_{i_q} P'_{j_q} P'_{k_q}, b^6) : 1 \leq q \leq n\}$$

$$W_3 = \{(T_q U_q, b^4) : 1 \leq q \leq n\}.$$

Let $W = W_1 \cup W_2 \cup W_3$.

We prove (**) as follows.

(1) Assume that there exists a homomorphism h in $\text{HOM}(V_\Psi, \{b\})$ such that, for every (α, β) in W , $h(\alpha) = \beta$.

Then let f be the valuation of V such that for every $q \in \{1, \dots, \ell\}$,

$f(P_q) = \text{false}$ if and only if $h(P_q) = b^2$. Since $W_1 \subseteq W$, f is well defined (and it follows that $f(P_q) = \text{true}$ if and only if $h(P_q) = b$).

Since $W_3 \subseteq W$, $1 \leq |h(T_q)| \leq 3$ for $1 \leq q \leq n$ and, because $W_2 \subseteq W$, this implies that $3 \leq |h(P'_{i_q} P'_{j_q} P'_{k_q})| \leq 5$ for $1 \leq q \leq n$. Thus for every

$q \in \{1, \dots, n\}$ either $|h(P'_{i_q})| = 1$ or $|h(P'_{j_q})| = 1$ or $|h(P'_{k_q})| = 1$ which implies that for every $q \in \{1, \dots, n\}$, $f((P'_{i_q} \vee P'_{j_q} \vee P'_{k_q})) = \text{true}$ and so Ψ is satisfied by f .

(2) Let Ψ be satisfiable and let f be a valuation of V which satisfies Ψ .

Let h be the homomorphism on V_Ψ^* defined by: for $1 \leq q \leq l$,

if $f(P_q) = \text{true}$ then $h(P_q) = b$ and $h(\overline{P}_q) = b^2$,

if $f(P_q) = \text{false}$ then $h(P_q) = b^2$ and $h(\overline{P}_q) = b$,

for $1 \leq q \leq n$,

if all three of $f(P'_{i_q})$, $f(P'_{j_q})$, $f(P'_{k_q})$ are equal *true*

then $h(T_q) = b^3$ and $h(U_q) = b$,

if only two of $f(P'_{i_q})$, $f(P'_{j_q})$, $f(P'_{k_q})$ are equal *true*

then $h(T_q) = b^2$ and $h(U_q) = b^2$,

if only one of $f(P'_{i_q})$, $f(P'_{j_q})$, $f(P'_{k_q})$ is equal *true*

then $h(T_q) = b$ and $h(U_q) = b^3$.

It follows directly from the definition of h that indeed for every (α, β)

in W , $h(\alpha) = \beta$.

Thus $(**)$ holds.

STEP 2.

Now given W from STEP 1 we construct x_Ψ, y_Ψ satisfying $(*)$ as follows

Let $W = \{(\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m)\}$ for some $m \geq 3$ and let $x_\Psi = \phi \alpha_1 \phi \dots \phi \alpha_m$ and

$y_\Psi = \phi \beta_1 \phi \dots \phi \beta_m$.

Note that if h is a homomorphism from $\text{alph } x_\Psi = V_\Psi \cup \{\phi\}$ into

$\text{alph } y_\Psi = B$ such that $h(x_\Psi) = y_\Psi$ then $h(\phi) = \phi$ (because both x_Ψ and y_Ψ

start with ϕ and $\phi \notin \text{alph } \beta_1$). Since $\#_{\phi} x_{\Psi} = \#_{\phi} y_{\Psi}$ this implies that

$$(***) \dots \left\{ \begin{array}{l} \text{there exists a homomorphism } h \text{ from } \text{alph } x_{\Psi} \text{ into } \text{alph } y_{\Psi} \\ \text{such that } h(x_{\Psi}) = y_{\Psi} \text{ if and only if there exists a homo-} \\ \text{morphism } g \text{ from } V_{\Psi} \text{ into } \{b\} \text{ such that } g(\alpha_q) = \beta_q \text{ for} \\ 1 \leq q \leq m. \end{array} \right.$$

But (***), (**) imply (*) and so the theorem holds.

Remark. If we change the definition of $\text{MATCH}(\Sigma, \Delta)$ to the definition of $\text{MATCH}_{\Lambda}(\Sigma, \Delta)$ by allowing arbitrary rather than only nonerasing homomorphisms then the theorem still remains true. That is we get the result:

"The membership in $\text{MATCH}_{\Lambda}(\Sigma, \Delta)$ is NP-complete." The main idea of the proof is the same and the only changes to be made are the following ones:

- (1) For every (α, β) in W_1 set $\beta = b$ (rather than $\beta = b^3$).
- (2) For every (α, β) in W_2 set $\beta = b^3$ (rather than $\beta = b^6$).
- (3) For every (α, β) in W_3 set $\beta = b^2$ (rather than $\beta = b^4$).
- (4) Given a homomorphism h "satisfying" W set the valuation f of V by:

$f(P_q) = \text{true}$ if and only if $h(P_q) = b$.

- (5) Given a valuation f of V satisfying Ψ set the homomorphism h by

if $f(P_q) = \text{true}$ then $h(P_q) = b$ and $h(\bar{P}_q) = \Lambda$,

if $f(P_q) = \text{false}$ then $h(P_q) = \Lambda$ and $h(\bar{P}_q) = b$,

if all three of $f(P'_{i_q})$, $f(P'_{j_q})$, $f(P'_{k_q})$ are equal *true*

then $h(T_q) = \Lambda$ and $h(U_q) = b^2$,

if only two of $f(P'_{i_q})$, $f(P'_{j_q})$, $f(P'_{k_q})$ are equal *true*

then $h(T_q) = b$ and $h(U_q) = b$,

if only two of $f(P'_{i_q})$, $f(P'_{j_q})$, $f(P'_{k_q})$ are equal *true*

then $h(T_q) = b^2$ and $h(U_q) = \Lambda$.

(6) In *STEP 2* define

$$x_{\Psi} = \textcircled{\alpha_1} \textcircled{\alpha_1} \textcircled{\alpha_1} \dots \textcircled{\alpha_m} \textcircled{\alpha_1} \textcircled{\alpha_1} \textcircled{\alpha_1} \dots \textcircled{\alpha_m} \text{ and}$$

$$y_{\Psi} = \textcircled{\beta_1} \textcircled{\beta_1} \textcircled{\beta_1} \dots \textcircled{\beta_m} \textcircled{\beta_1} \textcircled{\beta_1} \textcircled{\beta_1} \dots \textcircled{\beta_m}.$$

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- [1] A. Aho, J. Hopcroft and J. Ullman, The Design and Analysis of Computer Algorithms, Addison-Wesley, Reading, Mass., 1974.

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